# An Affine Arithmetic Method to Solve the Stochastic Power Flow Problem Based on a Mixed Complementarity Formulation

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# Outline

- Introduction
- Background Review and Tools
  - Classical Power Flow Analysis
  - Self-Validated Computation Methods
- Developed Models and Tools
  - Optimization Formulation of the Power Flow Problem
  - Power Flow Using Affine Method
- Future Work

# Motivation

- Power system operations (power flow analysis, optimal power flow (OPF), economic dispatch and unit commitment models) are expected to be impacted significantly, due to the intermittent nature of renewable sources.

- A wide variety of probabilistic methods are reported to incorporate uncertainties in renewable sources.
  - Monte-Carlo Simulation (MCS)
  - Analytical Methods (Convolution methods)
  - Probabilistic Methods (PDF)
  - Self Validated Computation Methods (SVC)
    - Interval Arithmetic (IA)
    - Affine Arithmetic (AA)

# Literature Review: Probabilistic Power Flow

- Probabilistic power flow first paper (Borkowska, 1974)
  - Uses PDF of independent demand variables, in a dc model
  - Convolution method is used to obtain the PDF of the output data
- (Leite da Silva et al, 1984) considered correlation between input random variable, and (Allan et al) uses linearization techniques to extend the probabilistic power flow model to ac system.
- Other methods such as FFT transformation based convolution method, multi-linearization technique, technique of moments, and Cumulants method are also reported to enhance the PDF of the power flow solution.
- Interval method is used by (Alvarado et al, 1992) to provide strict bounds to the solution of the power flow problem.

# Research Objectives

- Develop an optimization based model of the classical power flow problem using complementarity conditions, to properly represent generator bus voltage control, including reactive power limits and voltage recovery processes.
- Develop an accurate and efficient AA-based power flow model to incorporate uncertainties in power systems.
- Develop an accurate and efficient AA-based OPF model to incorporate uncertainties in power systems.
  - Validate the AA-based operation system models with the MCS based method.
- Develop an AA based method to estimate the spinning reserve requirements in the presence of high DG penetration.
- Develop an AA-based method to incorporate the uncertainties in Optimal Transmission Line Switching solution.

# Power Flow Analysis Problem: Definition

- Provides network solutions such as voltage magnitudes and angles for a given set of operating conditions.
- Uses iterative methods to solve the system of nonlinear equations:
  - Gauss-Seidel
  - Newton-Raphson
  - Fast Decoupled Newton-Raphson

# Power Flow: Newton-Raphson Method

$$\Delta P(\delta, P_S, |V_D|, Q_G) = P_i - |V_i| \sum_{j=1}^{n} |V_j| \left(G_{ij} \cos \delta_{ij} + B_{ij} \sin \delta_{ij}\right) = 0 \quad \forall i$$

$$\Delta Q(\delta, P_S, |V_D|, Q_G) = Q_i - |V_i| \sum_{j=1}^{n} |V_j| \left(G_{ij} \sin \delta_{ij} - B_{ij} \cos \delta_{ij}\right) = 0 \quad \forall i$$

$$f(x) \approx f(x^k) + \frac{df(x^k)}{dx}\left(x^{k+1} - x^k\right) = 0$$

$$\Delta x^{k+1} = -\alpha \left[\frac{df(x^k)}{dx}\right]^{-1} f(x^k)$$

$$x^{k+1} = x^k - \alpha \left[\frac{df(x^k)}{dx}\right]^{-1} f(x^k)$$

# Proposed MCP Based Power Flow Model

- The NR method has convergence problem, when using flat start in large systems.
- The power flow problem is formulated as an MCP to take advantage of robust and efficient state-of-the-art NLP and MCP solvers.
- The solution of the proposed MCP model is shown to be accurate, flexible and robust.

$$\min \quad F(\varepsilon_p, \varepsilon_q) = \sum_i \left\{ {\varepsilon_p}_i^{\,2} + {\varepsilon_q}_i^{\,2} \right\}$$

$$\text{s.t.} \quad \Delta P_i(\delta, P_s, |V_D|, |V_G|, Q_G) - {\varepsilon_p}_i = 0 \qquad \forall i$$

$$\Delta Q_i(\delta, P_s, |V_D|, |V_G|, Q_G) - {\varepsilon_q}_i = 0 \qquad \forall i$$

$$|V_{G_i}| = |V_{Gi_0}| + V_{Ga_i} - V_{Gb_i} \qquad \forall i \in \{gen\}$$

$$0 \le \left(Q_{G_i} - Q_{G_i}^{min}\right) \perp V_{Ga_i} \ge 0 \quad \forall i \in \{gen\}$$

$$0 \le \left(Q_{G_i}^{max} - Q_{G_i}\right) \perp V_{Gb_i} \ge 0 \quad \forall i \in \{gen\}$$

$$|V_{G_i}|, V_{Ga_i}, V_{Gb_i} \ge 0 \qquad \forall i \in \{gen\}$$

# Complementarity Constraints

$$0 \le \left(Q_{G_i} - Q_{G_i}^{min}\right) \perp V_{Ga_i} \ge 0 \qquad \forall i \in \{gen\}$$
$$0 \le \left(Q_{G_i}^{max} - Q_{G_i}\right) \perp V_{Gb_i} \ge 0 \qquad \forall i \in \{gen\}$$

$$\left(Q_{G_i} - Q_{G_i}^{min}\right) V_{Ga_i} = 0 \qquad \forall i \in \{gen\}$$
$$\left(Q_{G_i} - Q_{G_i}^{min}\right) \ge 0 \qquad \forall i \in \{gen\}$$
$$V_{Ga_i} \ge 0 \qquad \forall i \in \{gen\}$$

$$\left(Q_{G_i}^{max} - Q_{G_i}\right) V_{Gb_i} = 0 \qquad \forall i \in \{gen\}$$
$$\left(Q_{G_i}^{max} - Q_{G_i}\right) \ge 0 \qquad \forall i \in \{gen\}$$
$$V_{Gb_i} \ge 0 \qquad \forall i \in \{gen\}$$

# NR as an MCP Solution Step

The solution of the MCP power flow model, using the generalized reduced gradient (GRG) method corresponds to the NR solution of the power flow equations for $\varepsilon_p = \varepsilon_q = 0$.

$$\begin{aligned} &\min \quad F(\varepsilon) \\ &\text{s.t.} \\ &f(z) = \begin{bmatrix} \Delta P(\delta, P_S, |V_D|, Q_G) \\ \Delta Q(\delta, P_S, |V_D|, Q_G) \end{bmatrix} = 0 \end{aligned}$$

$$x = \begin{bmatrix} \delta \\ P_S \\ |V_D| \\ Q_G \end{bmatrix}, \varepsilon = \begin{bmatrix} \varepsilon_p \\ \varepsilon_q \end{bmatrix}$$

**Predictor Step:** Assuming that there is a set of values satisfying constraints $f(z)$, $z^m = (x^m, \varepsilon^m)$; the next improved solution is:

$$z^{m+1} = z^m + \beta s^m$$

$$s^m = -MM^T \nabla_z F(\varepsilon^m)$$

$$M = \begin{bmatrix} -[D_x f(z^m)]^{-1} D_\varepsilon f(z^m) \\ I_{2N} \end{bmatrix}_{(2N+2N)\times 2N}$$

**Corrector Step:** The predicted value of $z^{m+1}$ should then be corrected to ensure it satisfies the constraints $f(z) = \mathbf{0}$. It is done by the following robust NR procedure to obtain a $z^* = (x^*, \varepsilon^{m+1})$ such that $f(z^*) = \mathbf{0}$:

$$x^{k+1} = x^k - \alpha\left[D_x f(x^k, \varepsilon^{m+1})\right]^{-1} f(x^k, \varepsilon^{m+1})$$

# NR as an MCP Solution Step

Rewriting the MCP to include all the power flow and MCP equality and inequality constraints:

$$\begin{aligned} \min \quad & F(\varepsilon) \\ \text{s.t.} \quad & g(\hat{x},\varepsilon) = \begin{bmatrix} f(x,y,\varepsilon) \\ h(x,y,\varepsilon) \end{bmatrix} = 0 \\ & \hat{g}(x,y) \geq 0 \end{aligned}$$

$$\hat{x} = \begin{bmatrix} x \\ - \\ y \end{bmatrix} = \begin{bmatrix} \delta \\ P_S \\ |V_D| \\ Q_G \\ \hline |V_G| \\ V_{Ga} \\ V_{Gb} \end{bmatrix}, \varepsilon = \begin{bmatrix} \varepsilon_p \\ \varepsilon_q \end{bmatrix}$$

**Predictor Step:** Assuming that there is a set of values satisfying constraints $f(z)$, $z^m = (x^m, y^m, \varepsilon^m)$; the next improved solution is:

$$z^{m+1} = z^m + \beta s^m$$

$$s^m = -MM^T \nabla_z F(\varepsilon^m)$$

$$M = \begin{bmatrix} -[D_{\hat{x}} f(z^m)]^{-1} D_\varepsilon f(z^m) \\ I_{2N} \end{bmatrix}_{(2N+2N+3N_G)\times(2N+3N_G)}$$

**Corrector Step:**

$$\hat{x}^{k+1} = \hat{x}^k - \alpha \left[ D_{\hat{x}} g(\hat{x}^k, \varepsilon^{m+1}) \right]^{-1} g(\hat{x}^k, \varepsilon^{m+1})$$

The corrector step, for $\varepsilon = 0$ can be considered basically a “new” NR solution procedure to solve the power flow problem that properly models the generator voltage controls, since it accounts for the generator reactive power limits and its terminal voltage recovery.

# Computational Efficiency

CONVERGENCE ITERATIONS FOR MCP POWER FLOW MODEL

| | 14-bus | 30-bus | 57- bus | 118-bus | 300-bus | 1211-bus |
|---|---|---|---|---|---|---|
| COINIPOPT | 6 | 39 | 35 | 36 | 109 | INFES |
| CONOPT | 28 | 18 | 56 | 96 | 455 | INFES |
| EMP | 28 | 18 | 56 | 81 | INFES | INFES |
| MINOS | 5 | 7 | 19 | 98 | INFES | INFES |
| KNITRO | 10 | 7 | 49 | 672 | 2,684 | INFES |
| SNOPT | 253 | 9,852 | 14,145 | INFES | INFES | INFES |
| PATH-NLP | 6 | 3 | 6 | 9 | 20 | 10 |

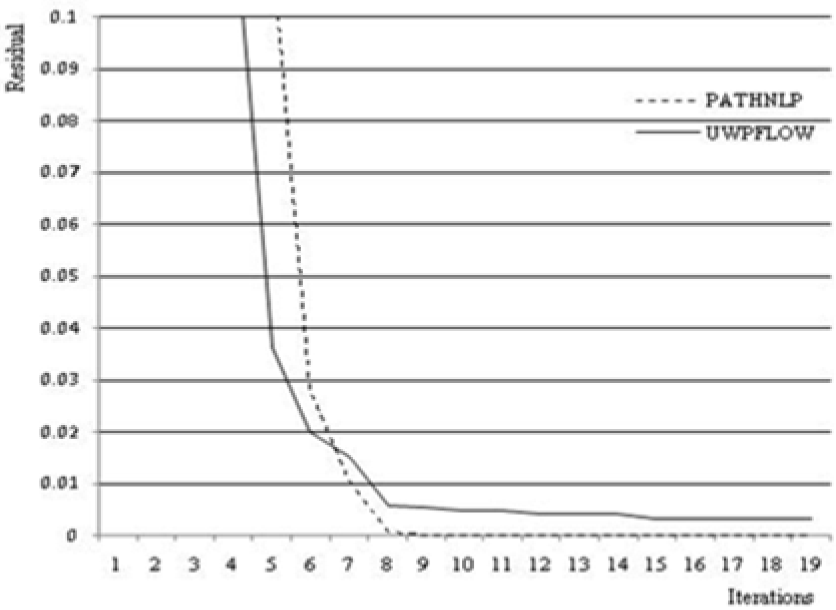
CONVERGENCE ITERATIONS FOR RECTANGULAR FORM MCP POWER FLOW MODEL

| | 14-bus | 30-bus | 57- bus | 118-bus | 300-bus | 1211-bus |
|---|---|---|---|---|---|---|
| COINIPOPT | 30 | 33 | 84 | 42 | 535 | 82 |
| CONOPT | 49 | 26 | 108 | 233 | 1438 | INFES |
| EMP | 50 | 26 | 155 | 202 | 1574 | INFES |
| MINOS | 7 | 7 | 8 | 6 | INFES | INFES |
| KNITRO | 18 | 20 | 10 | 19 | INFES | INFES |
| BARON | 127 | 494 | INFES | INFES | INFES | INFES |
| SNOPT | 10 | 14 | 23 | 10 | 234 | INFES |
| PATH-NLP | 3 | 3 | 6 | 9 | 12 | 12 |

# Computational Robustness

ITERATION COMPARISON OF THE PROPOSED MCP MODEL

| System | MCP-based Power Flow | Newton-Raphson Power Flow |
|---|---|---|
| IEEE 14-bus | 3 | 3 |
| IEEE 30-bus | 3 | 4 |
| IEEE 57-bus | 6 | 5 |
| IEEE 118-bus | 9 | 5 |
| IEEE 300-bus | 12 | 10 |
| 1200-bus | 12 | INFES |



# SVC Methods: Interval Analysis (IA)

- Self-validated range analysis technique, considering internal and external errors
- Providing the most conservative bounds

$$\hat{x} + \hat{y} = [\,\underline{x} + \underline{y}\,, \bar{x} + \bar{y}\,]$$

$$\hat{x} - \hat{y} = [\,\underline{x} - \bar{y}\,, \bar{x} - \underline{y}\,]$$

$$\hat{x}\,.\,\hat{y} = \left[\min\left\{\underline{x}.\underline{y}\,, \underline{x}.\bar{y}, \bar{x}.\underline{y}\,, \bar{x}.\bar{y}\right\}, \max\left\{\underline{x}.\underline{y}\,, \underline{x}.\bar{y}, \bar{x}.\underline{y}\,, \bar{x}.\bar{y}\right\}\right]$$

$$\frac{\hat{x}}{\hat{y}} = [\,\underline{x}\,, \bar{x}].[\,{}^{1}/_{\bar{y}}\,, {}^{1}/_{\underline{y}}]$$

- Disadvantages of IA:
  - Dependency problem
  - Overflow problem
  - Error explosion

# SVC Methods: Affine Arithmetic (AA)

- It is an enhanced model for self validated numerical modeling, in which the quantities of interests presented as affine forms of certain primitive variables.

- It keeps track of correlations between computed and input quantities

- Affine representation of a value:

$$\hat{x} = x_0 + x_1\varepsilon_1 + x_2\varepsilon_2 + \cdots + x_n\varepsilon_n$$

- Interval range:

$$[\hat{x}] = \left[x_0 - \sum_i |x_i| \,, x_0 + \sum_i |x_i|\right]$$

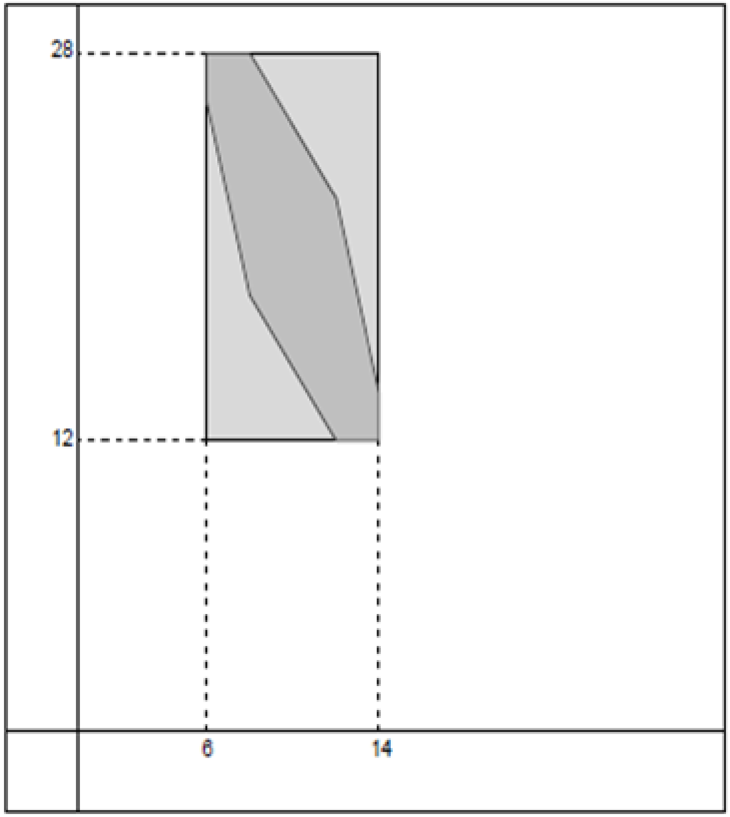
# AA vs. IA

$$\hat{x} = 10 + 2\varepsilon_1 + 1\varepsilon_2 \qquad - 1\varepsilon_4$$

$$\hat{y} = 20 - 3\varepsilon_1 \qquad + 1\varepsilon_3 + 4\varepsilon_4$$

$$\bar{x} = [6 \text{ -- } 14]$$

$$\bar{y} = [12 \text{ -- } 28]$$



# Basic Affine Operations

$$z = f(x, y) \rightarrow \hat{z} = \hat{f}(\hat{x}, \hat{y})$$

$$\hat{x} \pm \hat{y} = (x_0 \pm y_0) + (x_1 \pm y_1) + \cdots + (x_n \pm y_n)\varepsilon_n$$

$$\alpha\hat{x} = (\alpha x_0) + (\alpha x_1)\varepsilon_1 + (\alpha x_2)\varepsilon_2 + \cdots + (\alpha x_n)\varepsilon_n$$

$$\hat{x} \pm \varphi = (x_0 \pm \varphi) + x_1\varepsilon_1 + x_2\varepsilon_2 + \cdots + x_n\varepsilon_n$$

# Non-Affine Operations

$$z = f(x, y) \to \hat{z} = f(\hat{x}, \hat{y})$$

$$f^*(\varepsilon_1, \dots, \varepsilon_n) = x_0 y_0 + \sum_{i=1}^{n} (x_0\, y_i + y_0\, x_i)\, \varepsilon_i + \sum_{i=1}^{n} x_i \varepsilon_i \sum_{i=1}^{n} y_i \varepsilon_i$$

$$\hat{x}\, \hat{y} = x_0\, y_0 + \sum_{i=1}^{n} (x_0\, y_i + y_0\, x_i)\, \varepsilon_i + z_k\, \varepsilon_k$$

$$z_k = \sum_{i=1}^{n} |x_i| \sum_{i=1}^{n} |y_i|$$

# AA Power Flow: Bus Voltage Components in AA Form

$$\tilde{e}_i = e_{i,0} + \sum_{j \epsilon N} e_{i,j}^{P} \, \varepsilon_{P_j} + \sum_{j \epsilon N} e_{i,j}^{Q} \, \varepsilon_{Q_j}$$

$$\tilde{f}_i = f_{i,0} + \sum_{j \epsilon N} f_{i,j}^{P} \, \varepsilon_{P_j} + \sum_{j \epsilon N} f_{i,j}^{Q} \, \varepsilon_{Q_j}$$

# Affine Real and Reactive Power

Reactive power $\widetilde{P_i}$ and $\widetilde{Q_i}$ are calculated as follows:

$$\tilde{P} = \tilde{e}\,\tilde{I}_r + \tilde{f}\,\tilde{I}_{im}$$

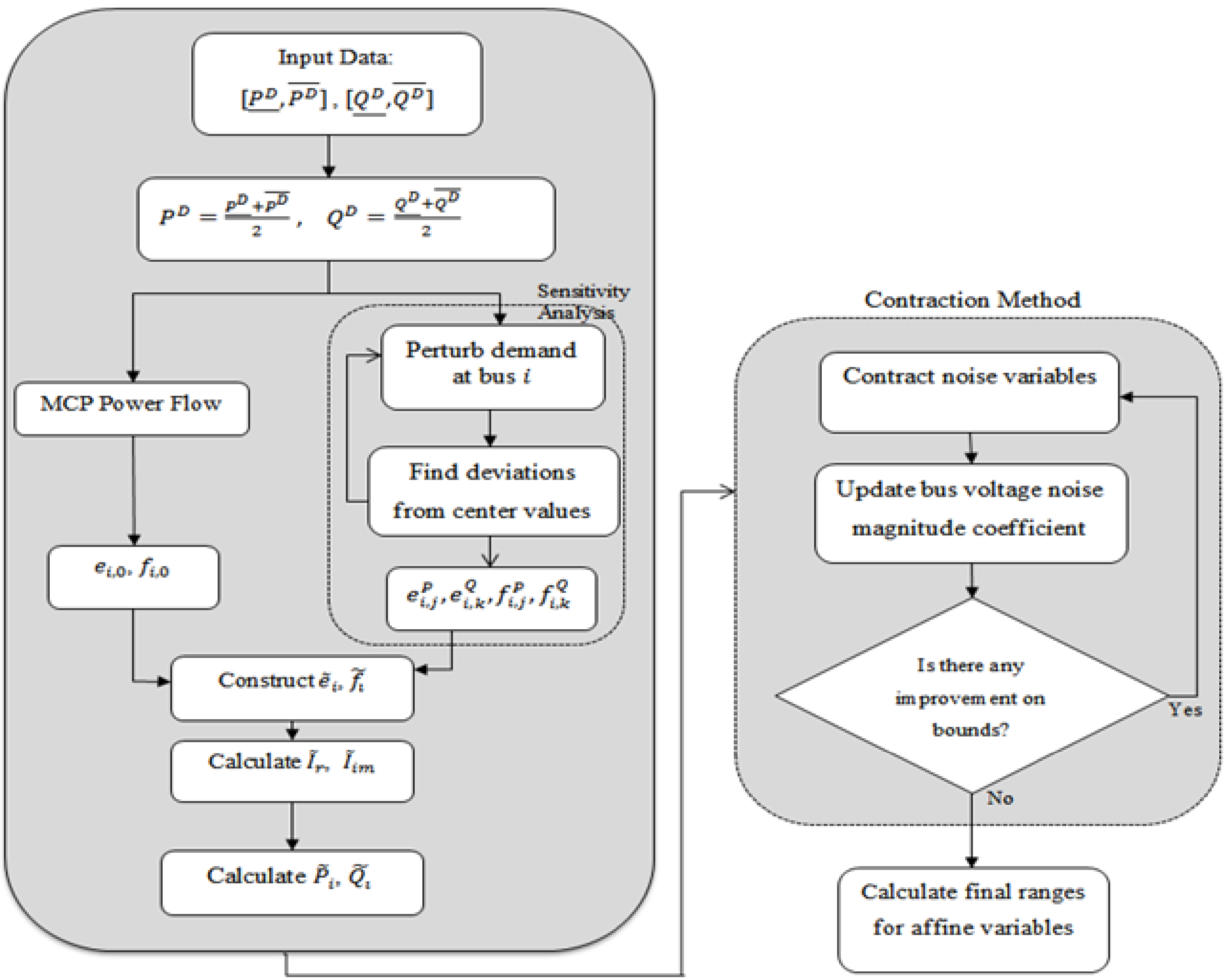
$$\tilde{Q} = \tilde{f}\,\tilde{I}_r - \tilde{e}\,\tilde{I}_{im}$$

$\widetilde{P_i}$ and $\widetilde{Q_i}$ have the following affine forms:

$$\widetilde{P_i} = P_{i,0} + \sum_j P_{i,j}^P\,\varepsilon_{P_j} + \sum_j P_{i,j}^Q\,\varepsilon_{Q_j} + P_i^T\,\varepsilon_{T_i}$$

$$\widetilde{Q_i} = Q_{i,0} + \sum_j Q_{i,j}^P\,\varepsilon_{Q_j} + \sum_j Q_{i,j}^Q\,\varepsilon_{Q_j} + Q_i^T\,\varepsilon_{T_i}$$

# AA Power Flow: Step-wise Procedure



# AA Power Flow: Contraction Method

$$\begin{bmatrix} \tilde{P}_1 \\ \cdots \\ \tilde{P}_N \\ \tilde{Q}_1 \\ \cdots \\ \tilde{Q}_{nPQ} \end{bmatrix} = \begin{bmatrix} P_{1,0} \\ \cdots \\ P_{N,0} \\ Q_{1,0} \\ \cdots \\ Q_{nPQ,0} \end{bmatrix} + \begin{bmatrix} P^P_{1,1} & P^P_{1,2} & \cdots & P^P_{1,N} \\ \cdots & \cdots & \cdots & \cdots \\ P^P_{N,1} & P^P_{N,2} & \cdots & P^P_{N,N} \\ Q^P_{1,1} & Q^P_{1,2} & \cdots & Q^P_{1,nPQ} \\ \cdots & \cdots & \cdots & \cdots \\ Q^P_{nPQ,1} & Q^P_{nPQ,2} & \cdots & Q^P_{nPQ,nPQ} \end{bmatrix} \begin{bmatrix} \varepsilon_{P_1} \\ \cdots \\ \varepsilon_{P_N} \end{bmatrix} + \begin{bmatrix} P^Q_{1,1} & P^Q_{1,2} & \cdots & P^Q_{1,N} \\ \cdots & \cdots & \cdots & \cdots \\ P^Q_{N,1} & P^Q_{N,2} & \cdots & P^Q_{N,N} \\ Q^Q_{1,1} & Q^Q_{1,2} & \cdots & Q^Q_{1,nPQ} \\ \cdots & \cdots & \cdots & \cdots \\ Q^Q_{nPQ,1} & Q^Q_{nPQ,2} & \cdots & Q^Q_{nPQ,nPQ} \end{bmatrix} \begin{bmatrix} \varepsilon_{Q_1} \\ \cdots \\ \varepsilon_{Q_n} \end{bmatrix} + \begin{bmatrix} P^T_1 \\ \cdots \\ P^T_N \\ Q^T_1 \\ \cdots \\ Q^T_{nPQ} \end{bmatrix} \begin{bmatrix} \varepsilon_{T_1} \\ \cdots \\ \varepsilon_{T_N} \\ \varepsilon_{T_{N+1}} \\ \cdots \\ \varepsilon_{T_{N+nPQ}} \end{bmatrix}$$

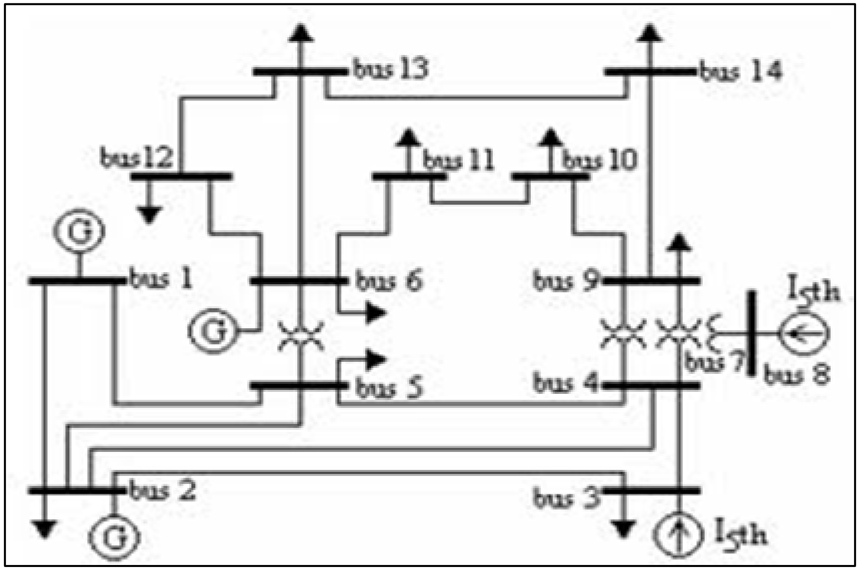
$$\tilde{f}(x) = A_0 + Ax + B_T x_T \quad \begin{cases} \overline{f}(x) = Ax^{Max} + B1 \\ \underline{f}(x) = Ax^{Min} + B2 \end{cases}$$

$\min \sum_i \varepsilon_i^{Min}$
s.t.
$-1 \leq \varepsilon_i^{Min} \leq 1$
$\underline{f}(x) \leq A\ \varepsilon_i^{Min} + B2 \leq \overline{f}(x)$

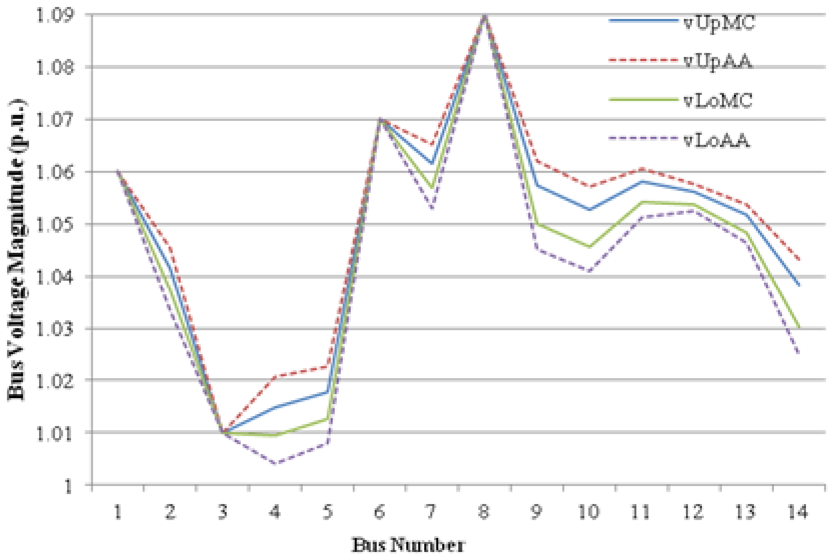
$\min \sum_i \varepsilon_i^{Max}$
s.t.
$-1 \leq \varepsilon_i^{Max} \leq 1$
$\underline{f}(x) \leq A\ \varepsilon_i^{Max} + B1 \leq \acute{f}(x)$

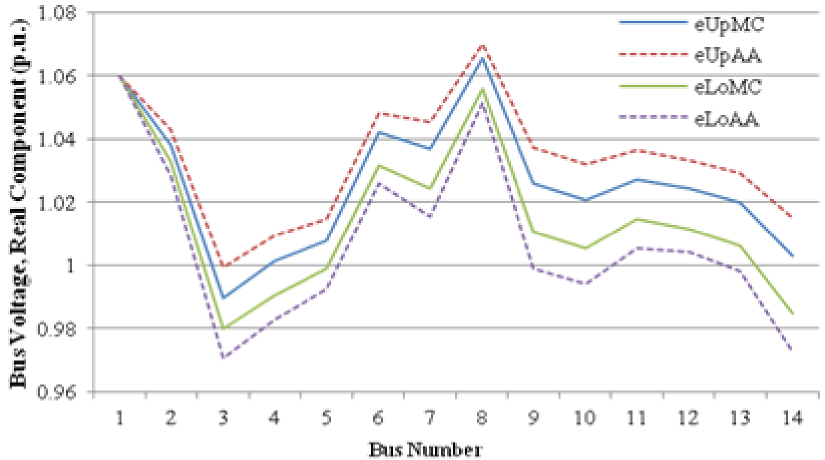
# Simulation Studies

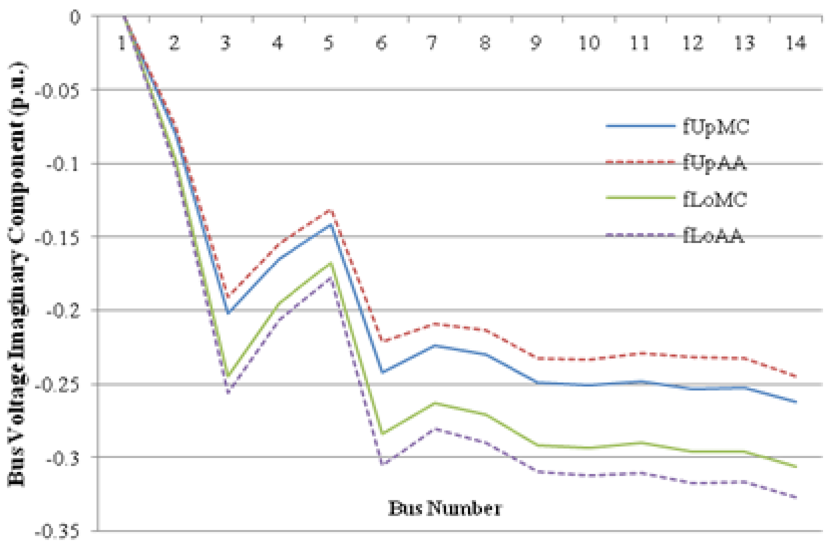
- AA is implemented in GAMS, for IEEE 14-bus test system.
- Results are compared with Monte-Carlo simulation.
- MCS uses:
  - 3000 iterations
  - Uniform distribution



# Results: AA-Based Bus Voltage Magnitude

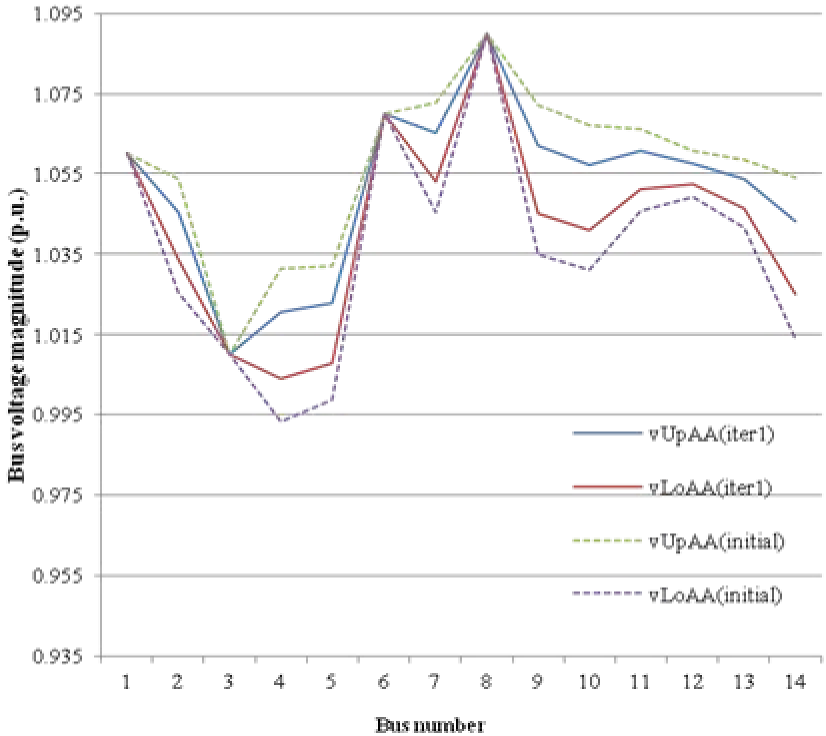


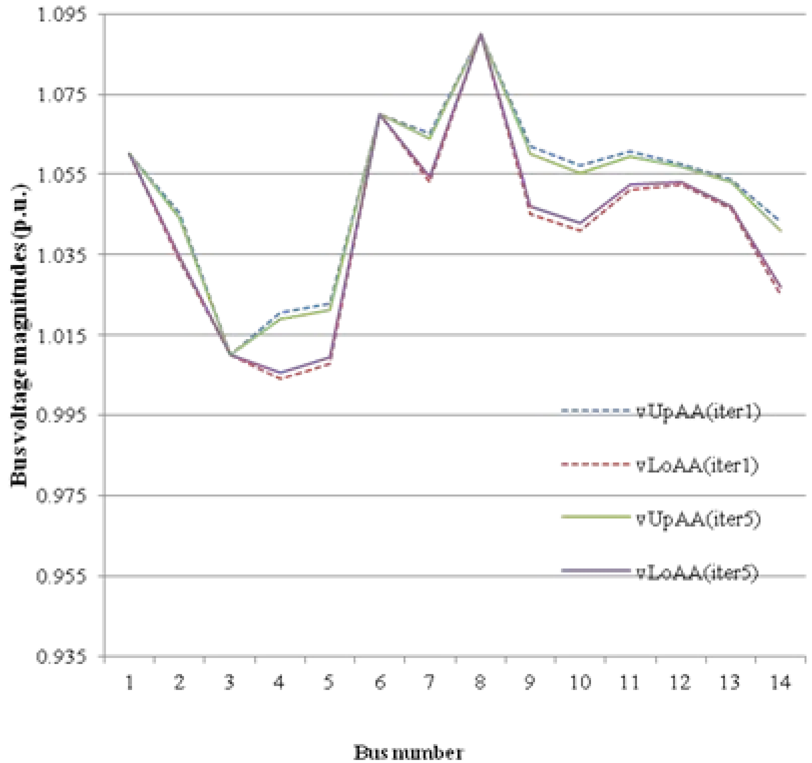



# Results: Contraction Method for Bus Voltage Magnitude

Contraction method after the 1st iteration

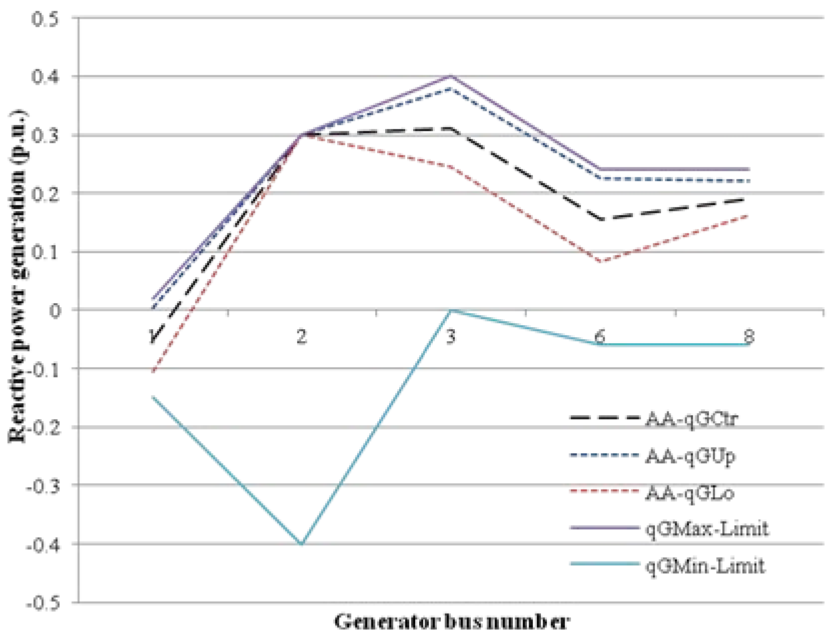



Contraction method after 5 iterations

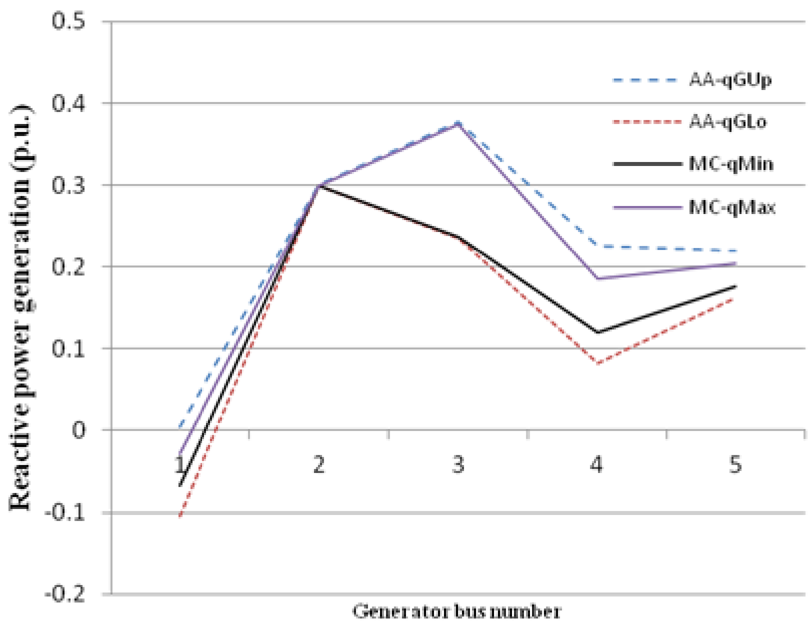


# Results-AA based reactive power limit

## Comparison with Given Limits




## Comparison with Monte-Carlo Limits



# Future Work

- Development of a computationally efficient and numerically accurate AA-based model to solve the stochastic OPF problem and the stochastic UC model with intermittent sources of energy such as wind and solar.
- Provide a detailed comparison of the AA-based solution with the other available solutions such as MCS in regards to its computational efficiency and numerical accuracy.
- Investigate the applications of the proposed AA-based OPF and UC methods to optimal spinning reserve requirements and optimal line switching.

# Thanks

# Research Papers/ Publications from this work


[1] M. Pirnia, C. Canizares and K. Bhattacharya, “Revisiting the Power Flow Problem Based on a Mixed Complementarity Formulation Approach” Submitted to IEEE Transactions on Power Systems, TPWRS-00787-2011, in revision.

[2] M. Pirnia, C. Canizares and K. Bhattacharya, A. Vaccaro, "An Affine Arithmetic Method to Solve the Stochastic Power Flow Problem Based on a Mixed Complementarity Formulation” Submitted to 2012 IEEE Power & Energy Society Annual General Meeting, San Diego, USA.