ACOPF: History and Formulations
(Alternating Current Optimal Power Flow)

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Outline

• ACOPF background
• Related problems
• Insights from 50 years of history
• ACOPF Formulations
ACOPF: What is it?

• Alternating Current Optimal Power Flow
• Optimization problem – optimize system dispatch subject to system and resource constraints
• Solved in different timeframes
  – Real time market: every 5 minutes
  – Day-ahead market: every 24 hours in hourly increments
  – Capacity market: annually for 3-5 years ahead
  – Transmission planning: annually for 10-15 future years
OPF

- OPF is a general term that describes a class of problems
  - ACOPF Includes
    - full power flow model and
    - system and resource constraints
  - DCOPF Assumes
    - voltage magnitudes constant,
    - voltage angles close to 0,
    - lossless (assume $R \ll X$) or lossy system
  - Decoupled OPF
    - Divides the ACOPF into linear subproblems
    - Iterates between the subproblems
ACOPF - basics

• Constraints
  – AC power flow equations
  – Equipment/operating/reliability constraints
    • Voltage, Current, Angle, Real Power, Reactive Power, Apparent Power

• Objective function
  – Maximize social welfare (if demand bids)
  – If demand is fixed, lowest system cost
Related Problems: Power Flow

- Power flow
  - Finds a feasible solution to the power flow equations, but is not an optimization problem
  - Formulated as AC, DC, and decoupled
  - Mismatch
  - Bus type classification
  - Need to match number of variables with number of equations to find solution
## Power flow – bus classification

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<tr>
<th>Bus Type</th>
<th>Fixed quantities</th>
<th>Variable quantities</th>
<th>Physical interpretation</th>
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</thead>
<tbody>
<tr>
<td>PV</td>
<td>Real power (P)</td>
<td>Reactive power (Q)</td>
<td>Generator</td>
</tr>
<tr>
<td></td>
<td>Voltage (V)</td>
<td>Angle (θ)</td>
<td></td>
</tr>
<tr>
<td>PQ</td>
<td>Real power (P)</td>
<td>Voltage (V)</td>
<td>Load, or generator with fixed output</td>
</tr>
<tr>
<td></td>
<td>Reactive power (Q)</td>
<td>Angle (θ)</td>
<td></td>
</tr>
<tr>
<td>Slack</td>
<td>Voltage (V)</td>
<td>Real power (P)</td>
<td>An arbitrarily chosen generator</td>
</tr>
<tr>
<td></td>
<td>Angle (θ)</td>
<td>Reactive power (Q)</td>
<td></td>
</tr>
</tbody>
</table>
Differences between power flow and OPF

- OPF is an optimization problem with constraints and objective function
  - The number of variables and constraints do not need to match
  - Bus type classifications are unnecessary and may introduce new constraints
- Power flow is a system of equations. It is often solved as a sort of optimization problem with the objective of minimizing “mismatch”
Related Problems: Economic Dispatch

- Economic dispatch
  - Optimization problem – minimize cost subject to generator output limits, overall constraint of generation = load + losses
  - Classic economic dispatch minimizes cost, but does not include network constraints
  - Security-constrained economic dispatch includes network constraints, usually formulated similar to DCOPF or decoupled OPF
History

- Early 20\textsuperscript{th} century:
  - ACOPF ‘solved’ by experienced engineers/operators using judgement, rules of thumb
  - Power flow problem - analog network analyzers
  - Economic dispatch – specialized slide rules
History – mid-century

• 1950’s - Digital solutions to the power flow
  – Ward and Hale – 1956
  – Iterative methods based on nodal admittance (Y) or nodal impedance (Z) matrix
  – Gauss-Seidel method

• 1960’s – Newton’s method for power flow
  – 1960’s – Tinney – sparsity techniques
History - 1962

• 50 years ago – Carpentier formulated ACOPF with some key insights:
  – A slack bus unnecessary in an optimization problem
  – Assume problem is suitably convex to apply the KKT conditions
• Based on google scholar, at least 236 papers have cited Carpentier’s original 1962 paper, even though it’s not available on internet
• 1968 – Dommel and Tinney – reiterate Carpentier’s insights, cited by at least 769
• ACOPF formulation has not changed significantly since 1962
History

• Several fairly comprehensive literature review papers since the 1980’s
• Stott and Alsac – decoupled OPF
• Literature since 1960’s has focused on different solution techniques, modeling improvements
• Challenges of the 1990’s persist today
  – Lack of uniform usage or problem definition
  – Local minima (is the problem really suitably convex to apply KKT?)
  – Lack of fast, robust, reliable nonlinear solution algorithms
Notation

• Assumption: balanced 3-phase steady-state operation
• When \( n \) and \( m \) are subscripts, they index buses;
• \( k \) indexes the transmission elements.
• When \( j \) is not a superscript, \( j = (-1)^{1/2} \);
• \( i \) is the complex current.
• When \( j \) is a superscript, it is the ‘imaginary’ part of a complex number.
• Matrices and vectors are upper case.
• Scalars and complex numbers are lower case.
Notation

- For column vectors $A$ and $B$ of length $n$, where $a_k$ and $b_k$ are the $k^{th}$ components of $A$ and $B$ respectively, the Hadamard product ‘·’ is defined so that $A·B = C$, where $C$ is a column vector also of length $n$, with $k^{th}$ component $c_k = a_kb_k$.
- The complex conjugate operator is $^*$ (superscript) and * (no superscript) is an optimal solution.
- **Indices and Sets**
  - $n, m$ are bus (node) indices; $n, m \in \{1, \ldots, N\}$ where $N$ is the number of buses. ($m$ is an alias for $n$)
  - $k$ is a three-phase transmission element with terminal buses $n$ and $m$.
  - $k \in \{1, \ldots, K\}$ where $K$ is the number of transmission elements between two buses; $k$ counts from 1 to the total number of transmission elements, and does not start over for each bus pair nm.
  - $K'$ is the number of a connected bus pairs ($K' \leq K$).
  - Unless otherwise stated, summations ($\sum$) are over the full set of indices.
Notation

- **Variables**
- $p_n$ is the real power injection (positive) or withdrawal (negative) at bus $n$
- $q_n$ is the reactive power injection or withdrawal at bus $n$
- $s_n = p_n + jq_n$ is the net complex power injection at bus $n$
- $p_{nmk}$ is the real power at bus $n$ to bus $m$ on transmission element $k$
- $q_{nmk}$ is the reactive power at bus $n$ to bus $m$ on transmission element $k$
- $\theta_n$ is the voltage phase angle at bus $n$
- $\theta_{nm} = \theta_n - \theta_m$ is the voltage phase angle difference from bus $n$ to bus $m$
Variables, continued

- $i$ is the current (complex phasor); $i_n$ is the current (complex phasor) injection (positive) or withdrawal (negative) at bus $n$ where $i_n = i_n^r + j i_n^i$

- $i_{nmk}$ is the current (complex phasor) injection (positive) or withdrawal (negative) flow in transmission element $k$ at bus $n$ (to bus $m$). $i_{nmk} = i_{nmk}^r + j i_{nmk}^i$

- $s_{nmk}$ is the apparent complex power injection (positive) or withdrawal (negative) into bus $n$ on transmission element $k$. $s_{nmk} = s_{nmk}^r + j s_{nmk}^i$

- $v_n$ is the complex voltage at bus $n$. $v_n = v_n^r + j v_n^i$
Notation

• **Variables, continued**
  • \( y_{nmk} \) is the complex admittance on transmission element \( k \) connecting bus \( n \) and bus \( m \) (If buses \( n \) and \( m \) are not connected directly, \( y_{nmk} = 0 \)); \( y_{n0} \) is the self-admittance (to ground) at bus \( n \).
  • \( y_{nm} \) is the complex admittance connecting bus \( n \) and bus \( m \) for all transmission elements \( k \) between buses \( n \) and \( m \).
  • \( V = (v_1, \ldots, v_N)^T \) is the complex vector of bus voltages; \( V = V^r + jV^i \)
  • \( I = (i_1, \ldots, i_N)^T \) is the complex vector of bus current injections; \( I = I^r + jI^i \)
  • \( P = (p_1, \ldots, p_N)^T \) is the vector of real power injections
  • \( Q = (q_1, \ldots, q_N)^T \) is the vector of reactive power injections
  • \( G \) is the \( N \)-by-\( N \) conductance matrix
  • \( B \) is the \( N \)-by-\( N \) susceptance matrix
  • \( Y = G + jB \) is the \( N \)-by-\( N \) complex admittance matrix
Notation

• **Functions and Transformations**

  • *Re*() is the real part of a complex number, for example, \( \text{Re}(i^n + j^n) = i^n \)
  
  • *Im*() is the real part of a complex number, for example, \( \text{Im}(i^n + j^n) = j^n \)
  
  • \( | | \) is the magnitude of a complex number, for example, \( |v_n| = \left[ (v^n_r)^2 + (v^n_j)^2 \right]^{1/2} \)
  
  • *abs*() is the absolute value function.
• Parameters

• $r_{nmk}$ is the resistance of transmission element $k$.
• $x_{nmk}$ is the reactance of transmission element $k$. $s_{k}^{max}$ is the thermal limit on apparent power over transmission element $k$ at both terminal buses.
• $\theta_{nm}^{min}, \theta_{nm}^{max}$ are the maximum and minimum voltage angle differences between $n$ and $m$.
• $p_{n}^{min}, p_{n}^{max}$ are the maximum and minimum real power for generator $n$.
• $q_{n}^{min}, q_{n}^{max}$ are the maximum and minimum reactive power for generator $n$.
• $C_{1} = (c_{1}^{1}, \ldots, c_{N}^{1})^{T}$ and $C_{2} = (c_{1}^{2}, \ldots, c_{N}^{2})^{T}$ are vectors of linear and quadratic objective function cost coefficients respectively.
Admittance Matrix

- Start with conductance \( G \), susceptance \( B \) and admittance \( Y \) matrices where \( gnm, bnm, \text{ and } ynm \) represent elements of the \( G \), \( B \), and \( Y \) matrices.

- Assume shunt susceptance negligible.

\[
\begin{align*}
g_{nmk} &= \frac{r_{nmk}}{(r_{nmk}^2 + x_{nmk}^2)} \quad \text{for} \quad n \neq m \\
b_{nmk} &= -\frac{x_{nmk}}{(r_{nmk}^2 + x_{nmk}^2)} \quad \text{for} \quad n \neq m \\
y_{nmk} &= g_{nmk} + j b_{nmk} \quad \text{for} \quad n \neq m \\
y_{nm} &= 0 \quad \text{for} \quad n = m \\
y_{nm} &= -\sum_k y_{nmk} \quad \text{for} \quad n \neq m \\
y_{nn} &= y_{n0} + \sum m \ y_{nm} \\
g_{nm} &= -\sum_k g_{nmk} \quad \text{for} \quad n \neq m \\
g_{nn} &= g_n + \sum m \ g_{nm} \\
b_{nm} &= -\sum_k b_{nmk} \quad \text{for} \quad n \neq m \\
b_{nn} &= b_n + \sum m \ b_{nm}
\end{align*}
\]
Transformers

- Y matrix above does not include transformer parameters.
- For an ideal in-phase transformer (assuming zero resistance in transformer windings, no leakage flux, and no hysteresis loss), the ideal voltage magnitude (turns ratio) is \( anmk = \frac{|vm|}{|vn|} \) and \( \theta_n = \theta_m \), where \( n \) is the primary side and \( m \) is the secondary side of the transformer.
- Since \( \theta_n = \theta_m \), \( anmk = \frac{|vm|}{|vn|} = \frac{vm}{vn} = -\frac{inm}{imn} \)
- The current-voltage equations for ideal transformer \( k \) between buses \( n \) and \( m \) are:
  - \( inmk = anmk^2ynmkvn - anmkynmkvm \)
  - \( imnk = -anmkynmkvn + ynmkvm \)
- For the phase shifting transformer (PAR) with a phase angle shift of \( \phi \),
  - \( \frac{vm}{vn} = tnmk = anmk e^{j\phi} \)
  - \( \frac{inm}{imn} = tnmk^* = -anmk e^{-j\phi} \)
- The current-voltage (IV) equations for the phase shifting transformer \( k \) between buses \( n \) and \( m \) are:
  - \( inmk = anmk^2ynmkvn - tnmk^*ynmkvm \)
  - \( imnk = -tnmkynmkvn + ynmkvm \)
Kirchhoff’s current law requires that the sum of the currents injected and withdrawn at bus \( n \) equal zero:

\[
i_n = \sum_k i_{nmk} \tag{2}
\]

If we define current to ground to be \( y_{n0}(v_n - v_0) \) and \( v_0 = 0 \), we have:

\[
i_n = \sum_k y_{nmk}(v_n - v_m) + y_{n0}v_n \tag{6}
\]

\[
i_{nmk} = y_{nmk}(v_n - v_m) = g_{nmk}(v_n^r - v_m^r) - b_{nmk}(v_n^i - v_m^i) + j(b_{nmk}(v_n^r - v_m^r) + g_{nmk}(v_n^i - v_m^i))
\]

\[
\begin{align*}
    i_{nmk}^r &= g_{nmk}(v_n^r - v_m^r) - b_{nmk}(v_n^i - v_m^i) \\
    i_{nmk}^i &= b_{nmk}(v_n^r - v_m^r) + g_{nmk}(v_n^i - v_m^i)
\end{align*}
\]

Current is a linear function of voltage. Rearranging,

\[
i_n = v_n(y_{n0} + \sum_k y_{nmk}) - \sum_k y_{nmk}v_m \tag{8}
\]
AC Power Flow Equations

- In matrix notation, the IV flow equations in terms of current (I) and voltage (V) in (8) are

\[ I = YV = (G + jB)(V^r + jV^i) = GV^r - BV^i + j(BV^r + GV^i) \]

(12)

where \( I^r = GV^r - BV^i \) and \( I^i = BV^r + GV^i \)

- In \( I \) and \( V \), the flow equations are linear

In another matrix format, (12) is

\[ I = (I^r, I^i) = Y(V^r, V^i)^T \]

or

\[ I = (I^r, I^i) = \begin{bmatrix} G & -B \\ B & G \end{bmatrix} \begin{bmatrix} V^r \\ V^i \end{bmatrix} \]

where \( Y = \begin{bmatrix} G & -B \\ B & G \end{bmatrix} \)
The traditional power-voltage power flow equations defined in terms of real power ($P$), reactive power ($Q$) and voltage ($V$) are:

- $S = P + jQ = \text{diag}(V)I^* = \text{diag}(V)[YV]^* = \text{diag}(V)Y^*V^*$

(16)

The power injections are

- $S = V \cdot I^* = (V^r + jV^i) \cdot (I^r - jI^i) = (V^r \cdot I^r + V^i \cdot I^i) + j(V^i \cdot I^r - V^r \cdot I^i)$

(18)

where

- $P = V^r \cdot I^r + V^i \cdot I^i$  
  (20)
- $Q = V^i \cdot I^r - V^r \cdot I^i$  
  (22)

The power-voltage power flow equations (16) and (18) are quadratic. The IV flow equations (12) are linear.
Constraints

• **Generator and load constraints**
  
  \[ P_{\text{min}} \leq P \leq P_{\text{max}} \quad \text{and} \quad Q_{\text{min}} \leq Q \leq Q_{\text{max}} \]

  - In terms of V and I, the injection constraints are:
    
    \[ P_{\text{min}} \leq V_r \cdot I_r + V_i \cdot I_i \leq P_{\text{max}} \]
    
    \[ Q_{\text{min}} \leq V_i \cdot I_r - V_r \cdot I_i \leq Q_{\text{max}} \]

• **Voltage constraints**
  
  \[(v_{\text{min}}^m)^2 \leq (v_r^m)^2 + (v_i^m)^2 \leq (v_{\text{max}}^m)^2\]

  - In matrix form:
    
    \[(V_{\text{min}})^2 \leq V_r \cdot V_r + V_i \cdot V_i \leq (V_{\text{max}})^2\]

• **Line flow thermal constraints**
  
  The apparent power at bus \(n\) on transmission element \(k\) to bus \(m\) is:

  \[ s_{nmk} = v_n \cdot i_{nmk}^* = v_n y_{nmk} (v_n - v_m) = v_n y_{nmk} v_n - v_n y_{nmk} v_m \]

  - The thermal limit on \(s_{nmk}\) is:
    
    \[ (s_{nmk}^r)^2 + (s_{nmk}^i)^2 = |s_{nmk}|^2 \leq (s_{\text{max}}^k)^2 \]

  - Or The thermal limit on \(i_{nmk}\) is:
    
    \[ (i_{nmk}^r)^2 + (i_{nmk}^i)^2 = |i_{nmk}|^2 \leq (i_{\text{max}}^k)^2 \]
Objective Functions

• The economically efficient objective function is to maximize social welfare. In the case of the OPF with fixed demand, that is the same as minimizing system cost.
  – Areas to explore – adding cost of reactive power, adding cost of switching

• Others:
  – Minimize losses
  – Minimize fuel cost
  – Minimize emissions
  – Minimize control actions
  – All of these other objective functions are redundant or sub-optimal in a ACOPF that models constraints and costs.
• Three formulations:
  – Polar P-Q (most common in literature)
  – Rectangular P-Q (less common in literature)
  – Rectangular I-V (new)
  – There are also a variety of hybrid formulations.
Formulations: Polar P-Q

- Network-wide objective function: \( \text{Min } c(S) \)  
- Network-wide constraints:
  - \( P_n = \sum_{mk} V_n V_m (G_{nmk} \cos \theta_{nm} + B_{nmk} \sin \theta_{nm}) \)  
  - \( Q_n = \sum_{mk} V_n V_m (G_{nmk} \sin \theta_{nm} - B_{nmk} \cos \theta_{nm}) \)  
    - These are quadratic-trigonometric equalities
  - \( V_{\text{min}} \leq V \leq V_{\text{max}} \)  
  - \( \theta_{\text{min}_{nm}} \leq \theta_n - \theta_m \leq \theta_{\text{max}_{nm}} \)
Formulations: Rectangular P-Q

- Network-wide objective function: \( \text{Min } c(S) \)
- Network-wide constraint: \( P + jQ = S = V \cdot I^* = V \cdot Y^* V^* \) \( (41) \)
- Bus-specific constraints
  - \( P_{\text{min}} \leq P \leq P_{\text{max}} \) \( (43) \)
  - \( Q_{\text{min}} \leq Q \leq Q_{\text{max}} \) \( (45) \)
  - \( (46')-(47') \) are replaced by:
    - \( V^r \cdot V^r + V^i \cdot V^i \leq (V_{\text{max}})^2 \) \( (46) \)
    - \( (V_{\text{min}})^2 \leq V^r \cdot V^r + V^i \cdot V^i \) \( (47) \)
    - \( (|s_{nmk}|)^2 \leq (s_{\text{max}}^k)^2 \) for all \( k \) \( (48) \)
  - \( (49') \) is replaced by:
    - \( \theta_{\text{min}} \leq \arctan(V^i_n/V^r_n) - \arctan(V^i_m/V^r_m) \leq \theta_{\text{max}} \) \( (49) \)
    - \( V^r \geq 0 \) \( (49.1) \)
Formulations: Rectangular I-V

- Network-wide objective function: \( \text{Min } c(S) \)  
  \( (50) \)
- Network-wide constraint: \( I = YV \)  
  \( (51) \)
- Bus-specific constraints:
  - \( P = V^r \cdot I^r + V^i \cdot I^i \leq P^{\text{max}} \)  
    \( (54) \)
  - \( r \leq P = V^r \cdot I^r + V^i \cdot I^i \)  
    \( (55) \)
  - \( Q = V^i \cdot I^r - V^r \cdot I^i \leq Q^{\text{max}} \)  
    \( (56) \)
  - \( Q^{\text{min}} \leq Q = V^i \cdot I^r - V^r \cdot I^i \)  
    \( (57) \)
  - \( V^r \cdot V^r + V^i \cdot V^i \leq (V^{\text{max}})^2 \)  
    \( (58) \)
  - \( (V^{\text{min}})^2 \leq V^r \cdot V^r + V^i \cdot V^i \)  
    \( (59) \)
  - \( (i^{\text{ink}})^2 \leq (i^{\text{max}})^2 \) for all \( k \)  
    \( (60) \)
  - \( \theta^{\text{min}}_{nm} \leq \arctan(v^r_n/v^r_m) - \arctan(v^i_n/v^i_m) \leq \theta^{\text{max}}_{nm} \)  
    \( (61) \)
- Can (60) make (61) redundant?
- \( V^r \geq 0 \)  
  \( (62) \)
## Comparison of Formulations

<table>
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<tr>
<th>Formulation</th>
<th>Polar PQ</th>
<th>Rectangular PQ</th>
<th>Rectangular IV</th>
</tr>
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<td>Network constraints</td>
<td>2N nonlinear quadratic and trigonometric equality constraints</td>
<td>2N quadratic equalities</td>
<td>2N linear equality constraints</td>
</tr>
<tr>
<td>Angle difference constraints</td>
<td><strong>Linear</strong></td>
<td>Nonconvex (arctan); <strong>Linear</strong> if replaced with current or apparent power constraint</td>
<td>Nonconvex (arctan); <strong>Linear</strong> if replaced with current or apparent power constraint</td>
</tr>
<tr>
<td>Bus constraints</td>
<td><strong>Linear</strong></td>
<td>Nonconvex quadratic inequalities</td>
<td>Locally quadratic, some nonconvex, some convex</td>
</tr>
</tbody>
</table>
Conclusions

• The ACOPF problem is inherently difficult due to nonconvexities, multipart nonlinear pricing, and alternating current.
• We do not yet have practical approaches to solving nonconvex problems.
• The ACOPF is a well-structured problem, and has developed during 50 years of research.
• The ACOPF is not a hypothetical problem – it is solved every 5 minutes through approximations and judgment.
• People have researched the ACOPF for 50 years, but there are still a lot of possibilities and ways to examine it.
• There is not yet a commercially viable full ACOPF. Since today’s solvers do not return the gap between the given and globally optimal solution.
• If we make a rough estimate that today’s solvers are on average off by 10%, and world energy costs are $400 billion, closing the gap by 10% is a huge financial impact.
Thank You

• Questions?

• Contact: mary.cain@ferc.gov